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Helical turbulence and absolute equilibrium

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The interaction of two pure helical (circularly polarized) velocity waves according to the incompressible Navier–Stokes equation produces modulation products of mixed helicity. In general, the interaction of waves of opposite helicity is stronger than that of waves with the same helicity. The inference is that strong net helicity depresses overall turbulent energy transfer. The conservation laws strongly inhibit energy transfer from higher to lower wavenumbers, when the helicity is large. The absolute equilibrium spectra of velocity and helicity for an inviscid flow system truncated at an upper wavenumber k_2 are

$$U(k) = 2\alpha/(\alpha^2-\beta^2k^2), \quad Q(k) = 2\beta k^2/(\alpha^2-\beta^2k^2),$$

where the velocity variance and helicity/unit volume are $\int U(k)\, d^3k$ and $\int Q(k)\, d^3k$, respectively. The temperature parameters α and β are constrained by $\alpha > 0$ and $|\beta k_2| < \alpha$. There are no analogues of the negative-temperature equilibrium states known for two-dimensional inviscid flow. It is argued that the inertial-range energy cascade in isotropic turbulence driven by helical input should not differ asymptotically from that of non-helical turbulence. The absolute equilibrium distributions suggest that, in contrast to the analogous two-dimensional situation, statistically steady helical input at middle wavenumbers should not produce a significant downward cascade of energy to lower wavenumbers.

1. Introduction

Turbulence with net helicity has received recent attention because of its probable importance in the generation of turbulent magnetic fields (Moffatt (1970, 1972); extensive further references are given in these papers). Such turbulence is also of basic theoretical importance because it emphasizes constants of motion whose existence has long been ignored. Moffatt (1969) showed that the total helicity of a bounded flow, defined as

$$\int \mathbf{u} \,.\, \boldsymbol{\omega} d^3x$$

($\mathbf{u}$ = velocity field, $\boldsymbol{\omega}$ = vorticity field), is an inviscid constant of motion because it measures the degree of linkage, or knottedness, of the vortex lines. Helicity conservation is thus a particular consequence of the Kelvin circulation theorem, which identifies an infinity of inviscid constants of motion: the circulations about all closed circuits that move with the fluid

The two quadratic constants of motion, kinetic energy and helicity, in inviscid three-dimensional flow suggest that there may be some analogies between three-dimensional turbulence with non-zero helicity and two-dimensional turbulence, where there are also two quadratic constants: energy and enstrophy. Certain of the possibilities have been discussed by Frisch, Lesieur and their colleagues (Lesieur, Frisch & Brissaud 1971; Brissaud *et al.* 1973). In the present paper, the analogy between the two kinds of flow is explored by examining the absolute inviscid equilibrium ensembles with net helicity, following an earlier treatment of absolute equilibrium ensembles in two-dimensional flow (Kraichnan 1967), and by noting some qualitative effects of the helicity-conservation constraint on the energy-cascade process in non-equilibrium states.

2. Interaction of two helical waves

A plane-polarized solenoidal velocity wave, for example

$$\mathbf{u}(\mathbf{x}) = (0, \sin(kx_3), 0),$$

has zero helicity; the vorticity and velocity vectors are perpendicular. But for the circularly polarized wave (Moffatt 1970)

$$\mathbf{u}(\mathbf{x}) = (\sin(kx_3), \cos(kx_3), 0), \tag{1}$$

we have

$$\boldsymbol{\omega}(\mathbf{x}) = \nabla\times\mathbf{u}(\mathbf{x}) = (k\sin(kx_3), k\cos(kx_3), 0), \tag{2}$$

so that

$$\mathbf{u}.\boldsymbol{\omega} = k|\mathbf{u}|^2.$$

By changing the sign of $u_1(\mathbf{x})$ in (1), we reverse the circular polarization and obtain a wave of opposite helicity The two helical waves provide an alternative to the usual Fourier decomposition into plane-wave components. In terms of the usual Fourier coefficients, defined by

$$\mathbf{u}(\mathbf{x}) = \sum_{\mathbf{k}} e^{i\mathbf{k}.\mathbf{x}}\mathbf{u}(\mathbf{k}), \tag{3}$$

where the sum is over all allowed $\mathbf{k}$ for a cyclical box of side L, (1) and (2) become

$$\mathbf{u}(\mathbf{k}) = (-\tfrac{1}{2}i, \tfrac{1}{2}, 0), \quad \mathbf{u}(-\mathbf{k}) = \mathbf{u}^*(\mathbf{k}), \tag{4}$$

$$\boldsymbol{\omega}(\mathbf{k}) = i\mathbf{k}\times\mathbf{u}(\mathbf{k}) = (-\tfrac{1}{2}ik, \tfrac{1}{2}k, 0). \tag{5}$$

Thus,

$$\boldsymbol{\omega}(\mathbf{k}).\mathbf{u}^*(\mathbf{k}) = k|\mathbf{u}(\mathbf{k})|^2. \tag{6}$$

This shows that (4) is a *pure* helicity wave, since it is trivial to show, from the definition of vorticity, that

$$|\boldsymbol{\omega}(\mathbf{k}).\mathbf{u}^*(\mathbf{k})| \leqslant |\mathbf{k}|\,|\mathbf{u}(\mathbf{k})|^2, \tag{7}$$

for any choice of $\mathbf{u}(\mathbf{k})$ that gives a real, solenoidal velocity field.

The incompressible Navier–Stokes equation may be written as

$$(\partial/\partial t+\nu k^2)\,\mathbf{u}(\mathbf{k}) = -i\sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}}\mathbf{k}.\mathbf{u}(\mathbf{q})\,[\mathbf{u}(\mathbf{p})]_{\perp\mathbf{k}}, \tag{8}$$

where ν is kinematic viscosity and $[\mathbf{u}(\mathbf{p})]_{\perp\mathbf{k}}$ is the vector projection of $\mathbf{u}(\mathbf{p})$ on the plane normal to $\mathbf{k}$. Now consider an inviscid flow in which the initial excitation is confined to pure helical waves at a single pair of wave vectors ($\pm\mathbf{p}$,

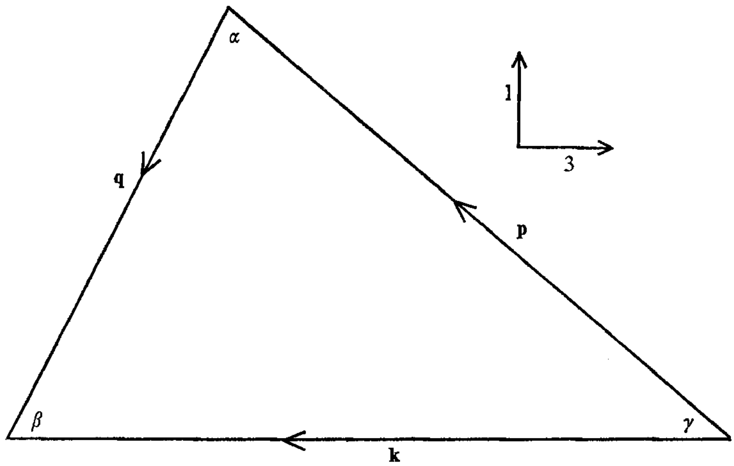

FIGURE 1. Geometry of the triad of interacting wave vectors.

$\pm\,\mathbf{q}$). Let $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{k} = \mathbf{p}+\mathbf{q}$ lie in the 1, 3 co-ordinate plane, with $\mathbf{k}$ pointing in the negative 3 direction, as shown in figure 1. Here α, β and γ are the interior angles of the triangle. Initially, (8) reduces to

$$\partial \mathbf{u}(\mathbf{k})/\partial t = -i\mathbf{k}.\mathbf{u}(\mathbf{q})\,[\mathbf{u}(\mathbf{p})]_{\perp\mathbf{k}} - i\mathbf{k}.\mathbf{u}(\mathbf{p})\,[\mathbf{u}(\mathbf{q})]_{\perp\mathbf{k}}. \tag{9}$$

We may take the two helical waves as

$$\mathbf{u}(\mathbf{p}) = (\tfrac{1}{2}i\cos\gamma, \tfrac{1}{2}, \tfrac{1}{2}i\sin\gamma), \quad \mathbf{u}(\mathbf{q}) = (\pm\tfrac{1}{2}i\cos\beta, \tfrac{1}{2}, \mp\tfrac{1}{2}i\sin\beta), \tag{10}$$

so that

$$\boldsymbol{\omega}(\mathbf{p}) = p\mathbf{u}(\mathbf{p}), \quad \boldsymbol{\omega}(\mathbf{q}) = \pm q\mathbf{u}(\mathbf{q}).$$

Thus, the upper signs in the expression for $\mathbf{u}(\mathbf{q})$ yield a $\mathbf{q}$ wave with the same sign of helicity as the $\mathbf{p}$ wave, while the lower signs give waves of opposite helicities. Substituting (10) into (9), we find

$$\partial \mathbf{u}(\mathbf{k})/\partial t = -\tfrac{1}{4}k[\mp i(\cos\gamma\sin\beta - \cos\beta\sin\gamma),\ \sin\gamma \pm \sin\beta, 0]. \tag{11}$$

Equation (11) shows, first of all, that the interaction of the pure helical $\mathbf{p}$ and $\mathbf{q}$ waves does *not* generate a $\mathbf{k}$ wave of pure helicity. This follows immediately from the fact that $|\dot{u}_1(\mathbf{k})| \neq |\dot{u}_2(\mathbf{k})|$. However, (11) shows an important asymmetry to the choice of same or different helicities for $\mathbf{p}$ and $\mathbf{q}$. For $\dot{u}_1(\mathbf{k})$, this choice merely affects the sign, but for $\dot{u}_2(\mathbf{k})$, the component normal to the plane of the wave vectors, the choice determines whether we have $\sin\gamma + \sin\beta$ or $\sin\gamma - \sin\beta$. The contrast is sharpest for the isosceles triangle $p = q$. Here the choice of same-signed helicity for $\mathbf{p}$ and $\mathbf{q}$ gives a null interaction, $\dot{u}(\mathbf{k}) = 0$, while helicities of different signs yield $\dot{u}_2(\mathbf{k}) \neq 0$. More generally we see that for $\beta, \gamma < \tfrac{1}{2}\pi$, which includes the case $p, q < k$, $|\dot{\mathbf{u}}(\mathbf{k})|$ is greater if the $\mathbf{p}$ and $\mathbf{q}$ waves have helicities of different sign than if they have the same sign. Since triangles of this kind may be expected to play an important, if not dominant, role in the usual isotropic energy cascade from lower to higher wavenumbers, we are led to an important qualitative inference: for a given energy-spectrum shape, overall energy transfer should be less for turbulence with maximal helicity,

$$\langle \boldsymbol{\omega}^*(\mathbf{k}).\mathbf{u}(\mathbf{k})\rangle = \pm k\langle|\mathbf{u}(\mathbf{k})|^2\rangle, \tag{12}$$

with the $\pm$ sign the same for all $\mathbf{k}$, than for reflexion-invariant turbulence. Moreover, it is clear that (12) will not survive under the Navier–Stokes equation, if it is imposed as an initial condition. Equation (11) suggests further that the asymmetry may be sufficient that the ordinary cascade process in reflexion-invariant turbulence is in fact dominated by interactions between wave pairs of opposite helicity. To confirm this conjecture, we must, of course, do more than examine initial behaviour. Further information could come from integration of closure approximations, or from direct computer simulation.

3. Absolute equilibium ensembles

Let us now consider the absolute statistical equilibrium ($\nu = 0$) of the truncated system obtained by retaining in (8) only those wavenumbers which fall in an interval (k_1, k_2). The total kinetic energy (divided by density) and total helicity may be written as

$$\tfrac{1}{2}(L/2\pi)^3 \sum_{\mathbf{k}} u_i(\mathbf{k})\, u_i^*(\mathbf{k})$$

and

$$i(L/2\pi)^3 \sum_{\mathbf{k}} \epsilon_{imj} k_m u_i(\mathbf{k})\, u_j^*(\mathbf{k}),$$

respectively, where the sum is over all retained wave vectors. Here ϵ_{imj} vanishes if any two indices are the same and equals ± 1 otherwise, according to whether an even or odd number of permutations is needed to bring the indices to the order 1, 2, 3.

The truncated system still conserves both energy and helicity. In fact, these quantities are conserved individually by each triad of interacting wave vectors $(\pm\mathbf{k}, \pm\mathbf{p}, \pm\mathbf{q})$. To see this, consider an instantaneous state in which any given set of wave vectors (and their negatives) have arbitrary amplitudes, subject only to the reality condition $\mathbf{u}(-\mathbf{k}) = \mathbf{u}^*(\mathbf{k})$, while the amplitudes at all other wave vectors are zero. Since, the energy and helicity expressions are quadratic and diagonal in the wave-vector amplitudes, the instantaneous rate of change of the energy and helicity in the instantaneously unexcited wave vectors is zero. Thus the overall conservation implies that the energy and helicity in the excited modes by themselves are conserved. But since the excited modes are chosen arbitrarily, and have arbitrary amplitudes, it follows that the conservation is an identity property of the coefficients, in the Navier–Stokes equation, that couple each individual triad of wave vectors. The detailed conservation properties can also be verified by direct calculation.

This truncated, inviscid system is an example of the class of dynamical systems whose variables y_i obey equations of the form

$$dy_i/dt = Y_i(y), \tag{13}$$

satisfy a generalized Liouville theorem,

$$\sum_i \partial Y_i(y)/\partial y_i \equiv 0, \tag{14}$$

and exhibit a constant of motion $K(y)$:

$$dK(y)/dt = \sum_i Y_i(y)\, \partial K(y)/\partial y_i \equiv 0. \tag{15}$$

The Y_i may be taken as the real and imaginary parts of the two independent components of the transverse vectors $\mathbf{u}(\mathbf{k})$; (14) then follows readily from (8) (Lee 1952), and K may be taken as

$$K = (L/2\pi)^3 \sum_{\mathbf{k}} [\alpha\delta_{ij} + i\beta\epsilon_{imj}k_m]\, u_i(\mathbf{k})\, u_j^*(\mathbf{k}), \tag{16}$$

a linear combination of energy and helicity.

It follows from (13)–(15) that $N \exp(-K)$, where N is a normalization factor, is a stable absolute equilibrium distribution, provided only that $\exp(-K)$ is integrable over y space. A consequence of the stability of the distribution (Kraichnan 1959) is the generalized fluctuation–dissipation theorem

$$\langle y_j(t)\, (\partial K/\partial y_i)_{t'}\rangle = g_{ji}(t, t'), \tag{17}$$

where $g_{ji}(t,t') = \langle \delta y_j(t)/\delta f_i(t')\rangle$ is the mean response to an infinitesimal forcing term f_i added to the right-hand side of (13). The stability property itself is that the distribution survives arbitrary, conservative couplings between the members of the ensemble.

For $t' = t$, (17) reduces to the generalized equipartition law

$$\langle y_j \partial K/\partial y_i\rangle = \delta_{ij}, \tag{18}$$

which also may be obtained directly from the form of the distribution $N \exp(-K)$ by partial integration (Tolman 1938). In our case, (18) can be written as

$$\langle u_j^*(\mathbf{k})\, \partial K/\partial u_i^*(\mathbf{k})\rangle = P_{ij}(\mathbf{k}), \tag{19}$$

where the transverse projection operator $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j/k^2$ serves to state explicitly that we constrain the velocity field to be solenoidal, a property preserved by (8). In (19), $\mathbf{u}$ and $\mathbf{u}^*$ are treated as formally independent while the differentiation is performed. The results are identical to those of the more proper procedure of taking real and imaginary parts as independent. Using (16), and carrying out the differentiation, we find, finally, that

$$(L/2\pi)^3 [\alpha\delta_{in} + i\beta\epsilon_{nmi}k_m]\, \langle u_n(\mathbf{k})\, u_j^*(\mathbf{k})\rangle = P_{ij}(\mathbf{k}) \tag{20}$$

is the equipartition law for the distribution $N \exp(-K)$.

The solution of (20) for the covariance is

$$(L/2\pi)^3 \langle u_i(\mathbf{k})\, u_j^*(\mathbf{k})\rangle = (\alpha^2 - \beta^2 k^2)^{-1} [\alpha P_{ij}(\mathbf{k}) + i\beta\epsilon_{imj}k_m], \tag{21}$$

a result which can be verified by substituting back into (20) and using the properties of ϵ_{imj}. The verification is easiest if one co-ordinate axis is taken parallel to $\mathbf{k}$ According to (21), the velocity variance and mean helicity per unit volume associated with $\mathbf{k}$ are, respectively, $(2\pi/L)^3\, U(k)$ and $(2\pi/L)^2 Q(k)$, where

$$U(k) = 2\alpha/(\alpha^2 - \beta^2 k^2), \tag{22}$$

$$Q(k) = 2\beta k^2/(\alpha^2 - \beta^2 k^2). \tag{23}$$

Again, we have used the properties of ϵ_{imj}. Now take the limit $L \to \infty$. Noting that

$$(2\pi/L)^3 \sum_{\mathbf{k}} \to \int d^3k,$$

we find for the total velocity variance and mean helicity per unit volume the values

$$U = 4\pi \int_{k_1}^{k_2} U(k)\, k^2\, dk = 8\pi\alpha \int_{k_1}^{k_2} (\alpha^2 - \beta^2 k^2)^{-1}\, k^2 dk, \tag{24}$$

$$Q = 4\pi \int_{k_1}^{k_2} Q(k)\, k^2\, dk = 8\pi\beta \int_{k_1}^{k_2} (\alpha^2 - \beta^2 k^2)^{-1}\, k^4\, dk. \tag{25}$$

In order for the formal results above to be valid, the distribution $N \exp(-K)$ must be integrable, which means that K, as given by (16), must be a positive form. The eigenvalues of the Hermitian matrix $\delta_{ij} + i\beta\epsilon_{imj} k_m$ are readily seen to be $\alpha, \alpha + \beta k$ and $\alpha - \beta k$, the first eigenvalue being irrelevant because we restrict the phase space to the transverse components of $\mathbf{u}(\mathbf{k})$. The conditions for integrability therefore are

$$\alpha > 0, \quad |\beta k_2| < \alpha. \tag{26}$$

The positivity of α means that there is no counterpart of the interesting negative-temperature, stable-equilibrium states that arise in inviscid two-dimensional systems (Kraichnan 1967).

Writing (24) and (26) as

$$U = 8\pi\alpha^{-1} \int (1 - r^2k^2)^{-1}\, k^2 dk, \quad Q = 8\pi\alpha^{-1} r \int (1 - r^2k^2)^{-1}\, k^4\, dk,$$

with $r = \beta/\alpha$, we see that U and $|Q|$ are both inversely proportional to α and both increase monotonically with increasing $|r|$, within the limits (26). However, the fractional increase in $|Q|$, as $|r|$ increases at any fixed α, exceeds that of U. This is a result, first, of the r factor outside the Q integral and, second, of the k^4 factor in the Q integrand, which emphasizes the higher k values where the denominator decreases more rapidly as $|r|$ increases. It follows that each realizable pair of U and Q values uniquely determines α and β. All positive U values are realizable, and the realizable range of Q is

$$|Q| < k_2 U. \tag{27}$$

As $|\beta| \to \alpha/k_2$, the spectra (22) and (23) become increasingly concentrated near k_2. The statistical form of (7) is

$$|Q(k)| \leqslant kU(k). \tag{28}$$

The equality is approached, in the limit, only for $k = k_2$. Thus, away from the singular limit, there are no absolute equilibrium states of maximal helicity. Suppose we have some initial state of maximal helicity, which then evolves into one of the absolute equilibrium states. The equilibrium state will have the same U and Q are the initial state, but the spectrum distribution will be different, and the equality in (28) will not hold at any k. The analysis in § 2 shows that there is no obvious constraint to prevent evolution into the equilibrium state, since we explicitly demonstrated there that maximal helicity is not preserved by the equations of motion.

4. Discussion

Our analysis suggests that the analogy between two-dimensional turbulence and three-dimensional helical turbulence is not a close one, despite the existence of two quadratic constants of motion in each case. The absolute equilibrium ensembles for the helical turbulence show none of the interesting structure associated with negative temperatures in the two-dimensional system. In the latter case, there were two kinds of extreme distributions: positive-temperature, high-enstrophy states with the excitation concerned near the upper cut-off wavenumber k_2; and negative-temperature, low-enstrophy states with excitation concentrated near the lower cut-off k_1 (Kraichnan 1967). The high-enstrophy states are analogous in form to the high-helicity states of the three-dimensional system, in which (27) is nearly an equality. But, in contrast to the low-enstrophy states, the low-helicity states $Q/U \to 0$ go over smoothly into the simple energy-equipartition equilibrium states obtained by considering only the energy constant of motion.

This difference is associated with a basic qualitative distinction between the nature of enstrophy and helicity. The enstrophy at a given wavenumber is determined by $U(k)$; for a given total energy, the total enstrophy can be changed only by changing the form of $U(k)$. On the other hand, the helicity contribution from given wavenumbers is independent of $U(k)$, subject only to (28).

The absolute equilibrium ensembles of course are very far from the actual states of viscous turbulence. Their value is in pointing to directions in which the actual, non-equilibrium states may be plausibly expected to transfer excitation. In this respect, the absence of absolute equilibrium states with energy peaked at low wavenumbers suggests that, in contrast to two-dimensional turbulence, we should not expect to find a simultaneous cascade of energy from middle wavenumbers to low wavenumbers and of helicity from middle wavenumbers to high wavenumbers. On the contrary, strong helicity can be expected to inhibit energy flow to lower wavenumbers, while the conservation laws do not constrain both energy and helicity from simultaneously cascading to higher wavenumbers. Thus, suppose that initially there is turbulence of maximal helicity confined to a narrow wavenumber band. Because of (28), downward transfer of energy without any upward transfer is impossible without violating the conservation laws.† However, upward transfer without any downward transfer is possible, since, as we saw in § 2, the maximal helicity condition is not preserved.

The plausible conclusion is that the inertial-range cascade of energy in isotropic helical turbulence should not differ qualitatively from that in ordinary reflexion-invariant turbulence. The arguments for local cascade, and an inertial-range law equal to or close to $-\frac{5}{3}$, seem neither weaker nor stronger. If there is local cascade, the conservation laws imply that, under the assumption of helical driving at low wavenumbers, the degree of helicity at higher wavenumbers, as measured by $|Q(k)|/[kU(k)]$, should decrease as one goes up in the inertial range.

† This fact is sufficient to infer the result, found by calculation in § 2, that two helical waves $\mathbf{p}$ and $\mathbf{q}$ of the same sign cannot transfer excitation to $\mathbf{k}$ if $p = q$ and $k < p$. An argument of analytic continuity extends the inference to $k > p = q$.

The major new qualitative effect with strongly helical turbulence promises to be the inhibited flow to lower wavenumbers. This is not important in isotropic flows, but may be important in other situations. Quantitatively, we expect, from the results of § 2, that the overall magnitude of energy transfer should be depressed by strong helicity.

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